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Connectivity Of Graph Of Mobius Function For '0'

K. K. Srimitra[1], D. Bharathi[2]
[1]*Research Scholar, Department of Mathematics, S. V. University, Tirupati.*
[2]*Professor, Department of Mathematics, S. V. University, Tirupati.*
Email: mitrakandala@yahoo.co.in[1], bharathikavali@gmail.com[2]

Abstract - In this paper, we consider the arithmetic graph called Graph of Mobius Function for '0', which is a simple graph defined on the arithmetic function as Mobius function. For this graph, we studied the connectivity including the vertex connectivity and edge connectivity. Also discussed block of a graph for this Graph of Mobius Function for '0'.

Index Terms - Mobius function, Graph of Mobius function, connectivity, Separable graph, Block.

1. INTRODUCTION

The graph of Mobius function for '0' was first introduced by Vasumathi [9]. This is a simple graph which relates the graph theory and Number theory. For the first time Nathanson [5] used Number theory in Graph theory. Since then so many researchers worked on these arithmetic graphs. For instance, in Cayley graphs, Madhavi and chalapathi [3] used the arithmetic functions like Euler totient function, Mangoldt function etc.

The graph of Mobius function for '0' is a simple arithmetic graph with vertex set V as the set of first n natural numbers, $V = \{1, 2, 3, \dots, n\}$ and two vertices $a, b \in V$ are adjacent if and only if the Mobius function value of the product of a, b is zero i.e., the edge set, $E = \{ab/\mu(ab) = 0, \ \ a, b \in V\}$. In 2016, Srimitra [6] studied this graph and stated some basic concepts like adjacency, degree of vertices and planarity. And in [7], he calculated the Independent dominating set, Independent domination number etc.

The Mobius function in [4, 8], is a well known function in Number theory and it is defined by $\mu(1) = 1$ and if $n > 1$, we write $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$ where, $p_1, p_2, p_3, \dots, p_k$ are prime numbers then the Mobius function, $\mu(n) = \begin{cases} (-1)^k, if \ \alpha_1 = \alpha_2 = \dots = \alpha_k = 1 \\ 0 \ \ , \qquad otherwise \end{cases}$.

The graphs used in this study are all simple graphs.

1.1 Preliminaries

The basic definitions and concepts which are needed for this study are taken from [1, 2].

A vertex v of a graph G is called a **cut vertex** of G if the number of components of $G - v$ is more than that of G i.e.,$\omega(G - v) > \omega(G)$. A **vertex cut** of G is a subset $V'$of V such that $G - V'$ is disconnected. The **connectivity** or **vertex connectivity** of a simple graph G, denoted by $k(G)$, is the smallest number of vertices in G whose deletion from G leaves either disconnected graph or trivial graph. Moreover, $k(G) = 0$ if and only if either G is trivial or disconnected.

A **cut edge** of a graph G is an edge e such that the number of components of $G - e$ is more than that of G i.e., $\omega(G - e) > \omega(G)$. An **edge cut** of G is a subset E' of E such that $G - E'$ is disconnected. The **edge connectivity** of a simple graph, denoted by $k'(G)$ or $k_e(G)$, is the smallest number of edges in G whose deletion from G leaves either disconnected graph or empty graph. Moreover, $k'(G) = 0$ if and only if either G is disconnected or empty graph.

A connected graph is said to be **Separable** if its vertex connectivity is 1. In other words, if a graph has cut vertex then that graph is called separable graph. A connected graph that has no cut vertices is called a **Block**.

2. MAIN THEOREMS

To calculate the vertex connectivity of the graph of Mobius function for '0', we use the following lemmas.

__**Lemma 2.1**__: - (Lemma 1 of [6]) If u (u≠1) is any vertex in $G(\mu_n^{(0)})$ then $\deg(u) =$
$$\begin{cases} n-1, & if \ u \ is \ square \ factor \\ n - 1 - \sum_{d/u} \mu(d) \left[\frac{n}{d}\right] + \sum^{n}_{\substack{v=2 \\ (u,v)=1 \\ \mu(v)=0}} (1), & other \ wise \end{cases}$$
and if u=1, then $\deg(u) = \sum^{n}_{\substack{v=2 \\ \mu(v)=0}} (1)$

__**Lemma 2.2**__: - (Theorem 2 of [6]) The minimum degree of the graph $G(\mu_n^{(0)})$ is $\delta = \sum^{n}_{\substack{v=2 \\ \mu(v)=0}} (1)$

__**Proof**__: - Let $G(\mu_n^{(0)})$ be a graph with n vertices.
From the Lemma – 2.1, the degree of the vertex 1 is given by $d(1) = \sum^{n}_{\substack{v=2 \\ \mu(v)=0}} (1)$

Let u be any vertex of $G(\mu_n^{(0)})$ such that $\mu(u) = 1$, $u \neq 1$

Then the vertex u is adjacent to every vertex x such that $\mu(x) = 0$, since $\mu(u.x) = 0$

Also the vertex u may be adjacent to other vertex y such that $\mu(y) \neq 0, (u,y) \neq 1$, where $y \leq n$

So, the neighborhood of the vertex u is $N(u) = \{x/\mu(x) = 0\} \cup \{y/\mu(y) \neq 0, (u,y) \neq 1\}$

Thus $d(u) = |N(u)| \geq \sum^{n}_{\substack{x=2 \\ \mu(x)=0}} (1)$

Then, $d(u) \geq d(1)$

Similarly, for any vertex w of $G(\mu_n^{(0)})$ such that $\mu(w) = -1$

Then the vertex w is adjacent to every vertex x such that $\mu(x) = 0$, since $\mu(w.x) = 0$

Also the vertex w may be adjacent to other vertex y such that $\mu(y) \neq 0, (w,y) \neq 1$, where$y \leq n$

So, the neighborhood of the vertex w is $N(w) = \{x/\mu(x) = 0\} \cup \{y/\mu(y) \neq 0, (w,y) \neq 1\}$

Thus $d(w) = |N(w)| \geq \sum^{n}_{\substack{x=2 \\ \mu(x)=0}} (1)$

Implies that $d(w) \geq d(1)$

Now it is clear that the degree of the vertex z is $d(z) = n - 1$ such that $\mu(z) = 0$

Thus the degree of the vertex 1 is the minimum degree of the graph $G(\mu_n^{(0)})$

Hence the minimum degree of the graph $G(\mu_n^{(0)})$ is $\delta\left(G(\mu_n^{(0)})\right) = d(1) = \sum^{n}_{\substack{v=2 \\ \mu(v)=0}} (1)$ ■

Theorem 2.3: - The vertex connectivity of the graph $G(\mu_n^{(0)})$ is $K\left(G(\mu_n^{(0)})\right) = \delta$, the minimum degree of the graph $G(\mu_n^{(0)})$.

Proof: - Consider the graph $G(\mu_n^{(0)})$

From the lemma 2.1 and the lemma 2.2, $\delta = d(1) = \sum^{n}_{\substack{u=2 \\ \mu(u)=0}} (1)$

i.e., Let $S = \{u/\mu(u) = 0\}$ then, $|S| = \delta$.

And from the lemma 2.2, the vertex '1' is adjacent to the vertices of S only in $G(\mu_n^{(0)})$

Implies, $G(\mu_n^{(0)}) - S$ remains the graph disconnected and the vertex '1' is isolated vertex.

Then, S is the vertex cut and $|S| = \delta$

Let K be the vertex connectivity of the graph $G(\mu_n^{(0)})$.

Implies $K \leq \delta \longrightarrow$ (1)

Let T $(\neq S)$ be the minimum vertex cut of the graph $G(\mu_n^{(0)})$, such that $|T| = K$.

If $K < \delta$ then, $|T| < |S|$

Let v be any vertex such that $v \in S$ and $v \notin T$, then from the theorem 1 of [7], v is adjacent to all the remaining vertices, so that the graph is connected.

Therefore, $K < \delta$ is not possible.

Thus from Eq. (1), $K = \delta$.

Hence, the vertex connectivity of $G(\mu_n^{(0)})$ is $K\left(G(\mu_n^{(0)})\right) = \delta$. ■

Corollary 2.4: - The edge connectivity of the graph $G(\mu_n^{(0)})$ is $K'\left(G(\mu_n^{(0)})\right) = \delta$, the minimum degree of the graph $G(\mu_n^{(0)})$.

Proof: - Consider the graph $G(\mu_n^{(0)})$

From the theorem 2.3, the vertex connectivity of the graph is the minimum degree of the graph.

i.e., $K = \delta \longrightarrow$ (1)

But we have from the theorem 3.1 of [1], for any graph G, $K \leq K' \leq \delta \longrightarrow$ (2), where K' is the edge connectivity.

From Eq. (1) & Eq. (2), we have, $\delta \leq K' \leq \delta$

i.e., $K' = \delta$

Hence the edge connectivity of the graph $G(\mu_n^{(0)})$ is $K'\left(G(\mu_n^{(0)})\right) = \delta$ ■

From figure 1, we can observe that in the graph $G(\mu_8^{(0)})$, there exists two vertices 4 and 8 such that

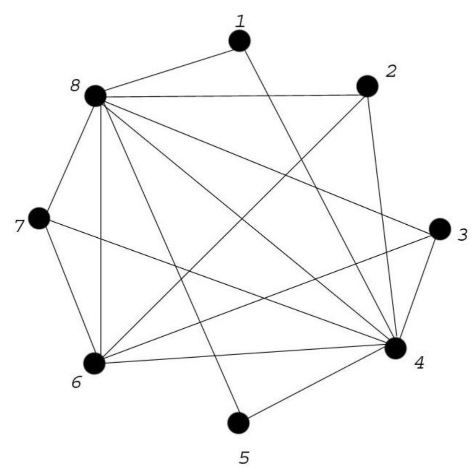

Figure 1: Graph of Mobius function for 0 with 8 vertices, $G(\mu_8^{(0)})$

$\mu(4) = \mu(8) = 0$. Also, the vertex connectivity and edge connectivity are equal to 2, the degree of the vertex 1 i.e., $k = k' = \delta = 2 = d(1)$.

Theorem 2.5: - The graph $G(\mu_n^{(0)}), 4 \leq n \leq 7$ is Separable graph.

Proof: - Consider the graph $G(\mu_n^{(0)}), 4 \leq n \leq 7$ with the vertex set $V = \{1, 2, 3, \ldots, n\}$

Since $4 \leq n \leq 7$, there exists only one $u \in V$ such that $\mu(u) = 0$.

i.e., $\{u\}$ is the only Independent Dominating Set in $G(\mu_n^{(0)}), 4 \leq n \leq 7$

And by the lemma 2.2, $\deg(1) = \delta = \sum^{n}_{\substack{v=2 \\ \mu(v)=0}} (1)$

Then $G(\mu_n^{(0)}) - \{u\}$ remains the vertex 1 as the isolated vertex

Since the vertex u is adjacent to all the remaining vertices, u is the only cut vertex in $G(\mu_n^{(0)}),\ 4 \le n \le 7$

Therefore the graph $G(\mu_n^{(0)}), 4 \le n \le 7$ has a cut vertex.

Hence the graph $G(\mu_n^{(0)}), 4 \le n \le 7$ is Separable graph. ■

Theorem 2.6: - The graph $G(\mu_n^{(0)}), n \ge 8$ is a Block.

Proof: - Consider the graph $G(\mu_n^{(0)}), n \ge 8$ with the vertex set $V = \{1, 2, 3, \ldots, n\}$

Since, $n \ge 8$, then there are some vertices such that whose Mobius function value is '0'

Let $S = \{u \in V / \mu(u) = 0\}$

Then all the vertices in $V - \{u\}$ are adjacent to $u, \forall u \in S$

Clearly$|S| \ge 2$

For instance, let $|S| = 2$ and $S = \{u, v\}$

Then the vertices u, v are adjacent to all the remaining vertices of $G(\mu_n^{(0)}), n \ge 8$

Let $x \in V - S$

In $G(\mu_n^{(0)}) - \{x\}$, all the vertices are connected to the vertex $u \in S$

Therefore, $G(\mu_n^{(0)}) - \{x\}$ is a connected graph for all $x \in V - S$

Thus, x is not a cut vertex of $G(\mu_n^{(0)}), n \ge 8,\ \forall x \in V - S$

Let $y \in S$

In $G(\mu_n^{(0)}) - \{y\}$, all the vertices are connected to all the vertices of $S - \{y\}$

Therefore, $G(\mu_n^{(0)}) - \{y\}$ is also a connected graph.

Thus y is not a cut vertex of $G(\mu_n^{(0)}), n \ge 8,\ \forall y \in S$

i.e., there is no cut vertex in $(V - S) \cup S = V$

Therefore, $G(\mu_n^{(0)}), n \ge 8$ has no cut vertex.

Hence $G(\mu_n^{(0)}), n \ge 8$ is a Block. ■

Remark: -The graph $G(\mu_n^{(0)}),\ n \le 3$ is neither seperable nor block. Because, all the vertices in this graph are isolated vertices.

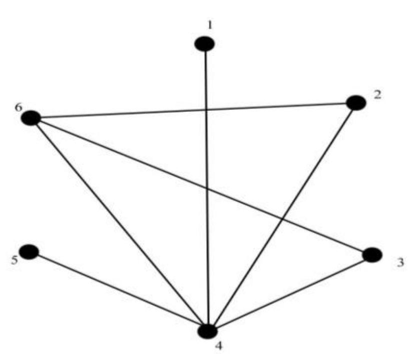

Figure 2: Graph of Mobius function for 0 with 6 vertices, $G(\mu_6^{(0)})$

In figure 2, the graph $G(\mu_6^{(0)})$ is a separable graph. If we remove the vertex 4 from the graph, then the vertex 5 becomes isolated vertex and thus the graph is disconnected.

3. CONCLUSION

In this paper, we studied about the graph of Mobius function for '0'. We mainly concentrated on the connectivity of the graph. It is obtained that the vertex connectivity and edge connectivity are equal to the minimum degree of the graph i.e., the degree of the vertex 1. Also found that the graph is separable if the number of vertices n lies between 4 and 7 and it is block if n is more than 7.

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